Uniform Current Dipoles and Loops
Part 1 - Theory

by
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Abstract – This tutorial paper discusses radiation from dipoles and loops carrying uniform current. It is shown that the radiation resistance for such a dipole is linearly dependent on its length. Then the radiation resistance of a square uniform current loop is calculated and found to be in agreement with the theory for circular loops put forward by Foster [1]. The realization of uniform current dipoles and loops is treated in a subsequent article (Part 2).

I. INTRODUCTION

Uniform current dipoles have never been regarded as practical, real world antennas in the literature. Uniform current loops, again, have been given brief descriptions in textbooks [2,3], usually in the small loop limit. In this two-part article, uniform current dipoles and loops are discussed theoretically, here, and as practically realized in Part 2. The intent of these tutorial articles is to show there is nothing abstract or non-physical about such antennas, and that the pattern & gain from a uniform current antenna can be enhanced over that of a sinusoidal distribution antenna.

In Section II the radiation resistance of a uniform current dipole is shown to depend linearly on length. Section III, for the first time, calculates the radiation resistance of square uniform current loops of any size. It is found that square loops are in good agreement with circular loops [1]. Section IV is a summary.

II. RADIATION RESISTANCE OF A UNIFORM CURRENT DIPOLE OF ANY LENGTH

We start by considering the simple case of an in-phase uniform distribution current element as in Figure 1, of overall length L_λ measured in wavelengths. The current is truncated at the element ends, which is quite realistic for a square loop where the current at a corner is carried away in an orthogonal direction. The current is taken to be $\vec{I}(z) = e^{-j\omega t}\,\hat{z}$ for z in the region $-L_\lambda/2 < z < L_\lambda/2$ and zero elsewhere.

The magnetic vector potential $\vec{A}$ at a point $P(r,\theta)$ far from the dipole is now developed. The scalar potential ϕ is not used, as it only contributes to near field terms.

We consider paired differential current elements $I\,dz_1$ and $I\,dz_2$, one located above the center of the dipole and the second located below the center of the dipole. The contribution to $\vec{A}$ at P from element 1 is

$$dA_1 = \frac{\mu_o}{4\pi} I \frac{e^{j\beta z\cos(\theta)}}{r} dz_1 \qquad (1)$$

The contribution from element 2 is

$$dA_2 = \frac{\mu_o}{4\pi} I \frac{e^{-j\beta z\cos(\theta)}}{r} dz_2 \tag{2}$$

where $|z_1| = |z_2| = |z|$. The center of the dipole is used as the phase reference. Both elements are z directed and the resultant $\vec{A}$ at P is also z directed. Summing the two contributions, we have

$$dA_z = dA_{z1} + dA_{z2} = \frac{\mu_o I}{4\pi r} 2\left[\frac{e^{j\beta z\cos\theta} + e^{-j\beta z\cos\theta}}{2}\right] dz = \frac{\mu_o I}{2\pi r}\cos\left(\beta z\cos\theta\right) \tag{3}$$

Taking the paired elements, we may integrate from the center of the dipole to the upper end to determine A in the far zone:

$$A_z(r,\theta) = \frac{\mu_o I}{2\pi r}\int_0^{L_\lambda/2} \cos\left(\beta z\cos\theta\right) dz \tag{4}$$

We take the $\hat{\theta}$ component of $\vec{A}$ at point P as

$$A_\theta(r,\theta) = \frac{\mu_o I}{2\pi r}\sin\theta \int_0^{L_\lambda/2} \cos\left(\beta z\cos\theta\right) dz \tag{5}$$

Let $u = \beta z\cos\theta$ and $du/dz = \beta\cos\theta$. With this substitution we have

$$A_\theta(r,\theta) = \frac{\mu_o I}{2\pi r}\frac{\sin\theta}{\beta\cos\theta}\int_0^{\beta\frac{L_\lambda}{2}\cos\theta} \cos u\, du \tag{6}$$

$$A_\theta(r,\theta) = \frac{\mu_o I}{2\pi\beta r}\tan\theta\sin\left(\beta\frac{L_\lambda}{2}\cos\theta\right) \tag{7}$$

E_θ in the far zone is simply

$$E_\theta = -j\omega A_\theta = \frac{-j\omega\mu_o I}{2\pi\beta r}\tan\theta\sin\left(\beta\frac{L_\lambda}{2}\cos\theta\right) \tag{8}$$

But $\omega\mu_o = \beta 120\pi$, and we have

$$E_\theta = \frac{-j\,60\,I}{r}\tan\theta\sin\left(\beta\frac{L_\lambda}{2}\cos\theta\right) \tag{9}$$

We may now calculate the total radiated power by integrating $E_\theta^* E_\theta$ over a large sphere in the far zone:

$$\begin{aligned} P &= \left(\frac{1}{2}\right)\frac{1}{120\pi}\int_0^{2\pi}\int_0^{\pi} E_\theta^* E_\theta\; r^2\sin\theta\, d\theta\, d\phi \\ &= \left(\frac{1}{2}\right)\frac{2\pi}{120\pi}\int_0^{\pi}\left(\frac{60\,I}{r}\right)^2\left[\tan\theta\sin\left(\beta\frac{L_\lambda}{2}\cos\theta\right)\right]^2 r^2\sin\theta\, d\theta \end{aligned} \tag{10}$$

This must equal the power delivered to the uniform current dipole, which is $(1/2)I^2\,R_{rad}$. Accordingly,

$$R_{rad} = 60\int_0^{\pi}\left[\tan\theta\sin\left(\beta\frac{L_\lambda}{2}\cos\theta\right)\right]^2\sin\theta\,d\theta \tag{11}$$

Since we are using a dipole length L_λ normalized in wavelengths, $\beta = 2\pi$.

Equation (11) is evaluated in Figures 2a,b. Figure 2a shows the radiation resistance for $L_\lambda = 0$ to 10 to be essentially linear with respect to the length. Figure 2b takes a closer look at the range $L_\lambda = 0$ to 1 to highlight the curvature of the locus as zero length is approached[1]. For the range $L_\lambda = 0$ to $\lambda/5$, the resistance is well represented [4] by

$$R_{rad} = 80\pi^2 L_\lambda^2 \tag{12}$$

III. Radiation Resistance of a Square Uniform Current Loop of Any Size

The radiation resistance for a circular uniform current loop of circumference C_λ (normalized to wavelength) is given in the classic treatment by Foster [1] as

$$R_{\substack{circular\\loop}} = 60\pi^2 C_\lambda^2\int_0^{\pi}\left[J_1\left(C_\lambda\sin\theta\right)\right]^2\sin\theta\,d\theta \tag{16}$$

Square loops have always been assumed to have a similar resistance.

Here we solve the square loop exactly, again using the vector potential. The square loop is modeled as 4 orthogonal current elements of normalized length L_λ so that $4L_\lambda$ equals the normalized perimeter P_λ. The voltage of each side of the loop for a current of $1\,e^{j\omega t}$ is given by

$$\tilde{V}_i = \begin{bmatrix} V_1\\ V_2\\ V_3\\ V_4\end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} & R_{14}\\ R_{21} & R_{22} & R_{23} & R_{24}\\ R_{31} & R_{32} & R_{33} & R_{34}\\ R_{41} & R_{42} & R_{43} & R_{44}\end{bmatrix}\bullet\begin{bmatrix}1\\1\\1\\1\end{bmatrix}e^{j\omega t} \tag{17}$$

Since the sides of the loop are identical, the total loop voltage is four times V_1.

$$V_{loop} = 4\left[R_{11}+R_{12}+R_{13}+R_{14}\right]e^{j\omega t} \tag{18}$$

Additionally, since $R_{12} = R_{14}$ by symmetry, we may write

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[1] When the radiation resistance needs to be evaluated on a calculator, the empirical formula below is always correct to within 7% for any L_λ.

$$R_{rad} = \left(120\pi\right)\frac{\pi}{2}L_\lambda\left(1 - e^{-\frac{\pi}{2}L_\lambda}\right)$$

$$V_{loop} = 4\left[R_{11} + 2R_{12} + R_{13}\right]e^{j\omega t} \tag{19}$$

Accordingly, there are only 3 resistance values to be determined, and R_{11} is given by (11).

The detailed expressions for R_{12} and R_{13} are given in the Appendix. The mathematics is straightforward but somewhat tedious. Since R_{12} and R_{13} depend on near field terms, the Appendix uses the full expression in $\vec{A}$ for $\vec{E}_{12}$ and $\vec{E}_{13}$, namely

$$\vec{E} = -j\omega\left[\vec{A} + \frac{1}{k^2}\nabla\nabla\bullet\vec{A}\right] \tag{20}$$

where the Lorenz gauge is used for the scalar potential ϕ.

The resulting resistance for the square loop is shown in Figure 3, along with the result of Foster [1] for a uniform current circular loop. The abscissa in the plot is the perimeter P_λ normalized with respect to wavelength. In the large loop regime, with $P_\lambda \geq 2$, the resistance varies as $60\pi^2 P_\lambda = 592\,P_\lambda$, in perfect agreement with Foster. In the small loop regime, with $P_\lambda \leq 1$, the resistance varies as $31170\,Area_\lambda^2$ [5], where $Area_\lambda$ is the loop area expressed in square wavelengths. Since the abscissa in Figure 3 is the normalized loop perimeter P_λ, the areas differ by a factor $4/\pi$ with the circular loop being the larger area. Accordingly, the ratio of the areas squared is $\left(4/\pi\right)^2 = 1.62$, which is the observed resistance ratio in the small loop regime as shown in Figure 4. Figures 3 and 4 also show $4R_{11}$; the deviations around $4R_{11}$ are the result of mutual coupling due to R_{12} and R_{13}.

IV. Summary

The radiation resistance for a uniform current dipole of any length is calculated, showing that the radiation resistance is essentially linear with respect to the length.

The vector potential method is used to analyze a square loop, with the result that square loops perform in theory just as circular loops.

A subsequent article discusses how these radiators may be realized. **–30-**

REFERENCES

1 Donald Foster, "Loop Antennas with Uniform Current", *Proc. IRE*, **32**, 603-607, October 1944.

2 John D. Kraus, Antennas, 2nd edition, pp.249-251, McGraw-Hill, 1988, New York.

3 Constantine A. Balanis, Antenna Theory, 2nd edition, pp.204-208, John Wiley & Sons, 1997, New York.

4 John D. Kraus, Antennas, 2nd edition, pg. 216, McGraw-Hill, 1988, New York.

5 *Ibid.*, pg. 251.

Appendix - Calculation of the Square Loop Mutual Resistances

We consider a square loop in the YZ plane as shown in Figure A1. The 4 sides of the loop, labeled 1, 2, 3, and 4 , are of identical length L_λ (expressed in wavelengths). The fields around the 4 elements cause the elements to interact, so that current on element j results in a voltage on element i. This effect may be expressed as the 4x4 matrix of self and mutual resistances $\left[R_{ij}\right]$ where each resistance value is defined as

$$R_{ij} = \frac{\text{voltage on element } i}{\text{current on element } j} \tag{A1}$$

with all other currents set to zero.

We consider the case where the loop carries a uniform in-phase current $I = e^{j\omega t}$. Due to the symmetry of the loop and the assumed uniform current, there are only 3 unique values for R_{ij}, these being

R_{11} = the element self resistance

R_{12} = the mutual resistance between adjacent sides of the loop

R_{13} = the mutual resistance between opposite sides of the loop

It should be clear that

$$R_{11} = R_{22} = R_{33} = R_{44}$$

$$R_{12} = R_{21} = R_{23} = R_{32} = R_{34} = R_{43} = R_{41} = R_{14}$$

$$R_{13} = R_{31} = R_{24} = R_{42}$$

R_{11} is calculated in the main text in equation (11).

Calculation of R_{12}

This calculation is valid for any two adjacent orthogonal sides of the loop. We use the notation in Figure A2 for the calculation of R_{12}. We also use equation (20) to calculate the voltage on element 1 due to the current in element 2. Since elements 1 and 2 are orthogonal, there is no direct contribution from $-j\omega\vec{A}$, and equation (20) reduces to

$$\vec{E} = \frac{-j\omega}{k^2}\nabla\nabla\bullet\vec{A} \tag{A2}$$

Since $A_{x2} = A_{z2} = 0$ for the potential of element 2, the electric field z-component at element 1 due to the current on element 2 is

$$E_{z12} = \frac{-j\omega}{k^2}\frac{\partial^2 A_{y2}}{\partial y_1 \partial z_1} \tag{A3}$$

where

$$A_{y2} = \frac{\mu_o}{4\pi}\int_{y_2=0}^{y_2=L_\lambda} I\frac{e^{-j\beta r_{12}}}{r_{12}}dy_2 \tag{A4}$$

and the explicit expression for r_{12} is

$$r_{12} = \sqrt{\left(z_1 - z_2\right)^2 + \left(y_1 - y_2\right)^2} \tag{A5}$$

The subscripts of the coordinates denote the element where the observation and integration points lie.

Placing (A4) into (A3) and reversing the order of differentiation

$$E_{z12} = -\frac{j\omega\mu_o I}{4\pi k^2}\frac{\partial}{\partial z_1}\int \frac{\partial}{\partial y_1}\left[\frac{e^{-j\beta r_{12}}}{r_{12}}\right] dy_2 \tag{A6}$$

The partial derivative with respect to y_1 equals

$$\frac{-(1+j\beta r_{12})\,e^{-j\beta r_{12}}}{r_{12}^2}\frac{\partial r_{12}}{\partial y_1} \tag{A7}$$

which along element 1 ($y_1 = 0$) is

$$\frac{-(1+j\beta r_{12})\,e^{-j\beta r_{12}}}{r_{12}^2}\left(\frac{-y_2}{r_{12}}\right) \tag{A8}$$

(A6) becomes

$$E_{z12} = -\frac{j\omega\mu_o I}{4\pi k^2}\frac{\partial}{\partial z_1}\int_{y_2=0}^{y_2=L_\lambda} \frac{(1+j\beta r_{12})\,e^{-j\beta r_{12}}}{r_{12}^3}\, y_2\, dy_2 \tag{A9}$$

The voltage along element 1 due to the current in element 2 is the integral of E_z along element 1:

$$V_{12} = \int_{z_1=0}^{z_1=L_\lambda} E_{z_{12}}\, dz_1 = -\frac{j\omega\mu_o I}{4\pi k^2}\left[\int_{y_2=0}^{y_2=L_\lambda} \frac{(1+j\beta r_{12})\,e^{-j\beta r_{12}}}{r_{12}^3}\, y_2\, dy_2\right]_{z_1=0}^{z_1=L_\lambda} \tag{A10}$$

$$= -\frac{j\omega\mu_o I}{4\pi k^2}\left\{\int_{y_2=0}^{y_2=L_\lambda} \frac{(1+j\beta\sqrt{y_2^2+L_\lambda^2})\,e^{-j\beta\sqrt{y_2^2+L_\lambda^2}}}{\left(y_2^2+L_\lambda^2\right)^{\frac{3}{2}}}\, y_2\, dy_2 - \lim_{\varepsilon\to 0}\int_{y_2=\varepsilon}^{y_2=L_\lambda} \frac{(1+j\beta y_2)\,e^{-j\beta y_2}}{y_2^3}\, y_2\, dy_2\right\}$$

The second integral is well defined in the limit as $\varepsilon \to 0$. The leading coefficient for the bracketed expression is equal to $-j15/\pi$ since $\omega\mu_o = 120\pi\,\beta$ and $k = \beta = 2\pi$ for the normalization used. Accordingly, $R_{12} = \mathrm{Re}(V_{12}/I)$.

CALCULATION OF R_{13}

We use the notation in Figure A3 for the calculation of R_{13}, using equation (20) to calculate the voltage on element 1 due to the current in element 3. Since elements 1 and 3 are parallel, both terms in equation (20) must be considered. The first term is straightforward to calculate.

$$E_{z13}(1) = -j\omega A_{z3} = -j\omega \frac{\mu_o I}{4\pi} \int_{z_3=0}^{z_3=L_\lambda} \frac{e^{-j\beta r_{13}}}{r_{13}} dz_3 \qquad \text{(A11)}$$

where $r_{13} = \sqrt{L_\lambda^2 + (z_3 - z_1)^2}$. Accordingly,

$$V_{13}(1) = \frac{-j\omega\mu_o I}{4\pi} \int_{z_1=0}^{z_1=L_\lambda} \int_{z_3=0}^{z_3=L_\lambda} \frac{e^{-j\beta\sqrt{L_\lambda^2+(z_3-z_1)^2}}}{\sqrt{L_\lambda^2 + (z_3 - z_1)^2}} dz_3\, dz_1 \qquad \text{(A12)}$$

The second term uses the same formalism as in the previous section:

$$E_{z13}(2) = \frac{-j\omega\mu_o I}{4\pi k^2} \frac{\partial}{\partial z_1} \int \frac{\partial}{\partial z_1} \left[\frac{e^{-j\beta r_{13}}}{r_{13}} \right] dz_3 \qquad \text{(A13)}$$

Then, evaluating the partial derivative and integrating

$$V_{13}(2) = \frac{-j\omega\mu_o I}{4\pi k^2} \left[\int_{z_3=0}^{z_3=L_\lambda} \frac{\partial}{\partial z_1} \left(\frac{e^{-j\beta r_{13}}}{r_{13}} \right) dz_3 \Big|_{z_1=L_\lambda} - \int_{z_3=0}^{z_3=L_\lambda} \frac{\partial}{\partial z_1} \left(\frac{e^{-j\beta r_{13}}}{r_{13}} \right) dz_3 \Big|_{z_1=0} \right] \qquad \text{(A14)}$$

The integrals are odd around $z_1 = L_\lambda / 2$ and may be evaluated at $z_1 = 0$ with twice the value.

$$V_{13}(2) = \frac{-j\omega\mu_o I}{4\pi k^2} (-2) \int_{z_3=0}^{z_3=L_\lambda} \frac{\partial}{\partial z_1} \left(\frac{e^{-j\beta r_{13}}}{r_{13}} \right) dz_3 \Big|_{z_1=0} \qquad \text{(A15)}$$

Substituting the partial derivative

$$V_{13}(2) = \frac{-j\omega\mu_o I}{4\pi k^2} (-2) \int_{z_3=0}^{z_3=L_\lambda} \frac{\left(1 + j\beta\sqrt{L_\lambda^2 + z_3^2}\right) e^{-j\beta\sqrt{L_\lambda^2+z_3^2}}}{\left(L_\lambda^2 + z_3^2\right)^{\frac{3}{2}}} z_3\, dz_3 \qquad \text{(A15)}$$

And finally

$$R_{13} = \frac{\mathrm{Re}\left[V_{13}(1) + V_{13}(2)\right]}{I} \qquad \text{(A16)}$$

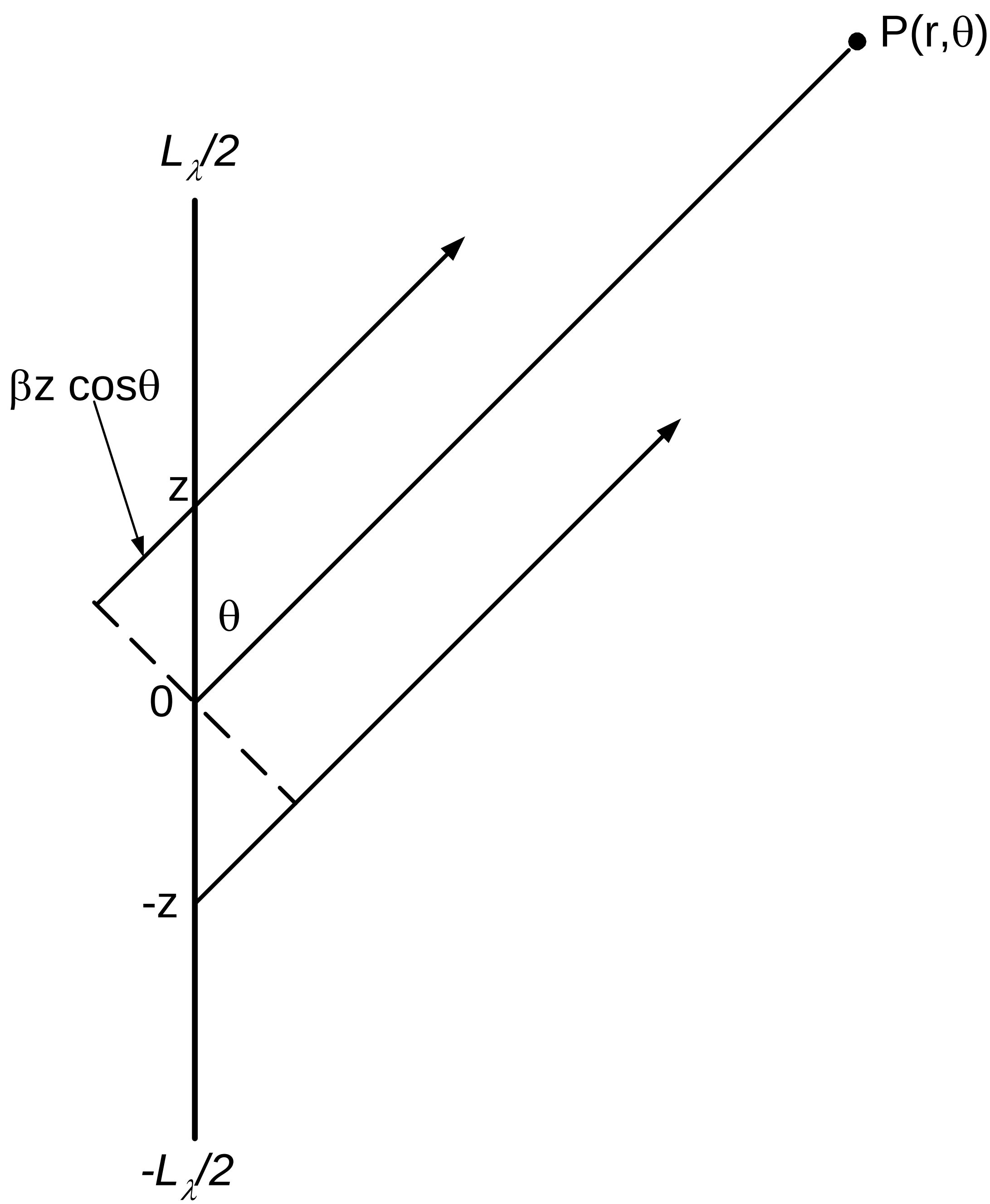

Figure 1 – Geometry of current element for far field analysis with the vector potential.

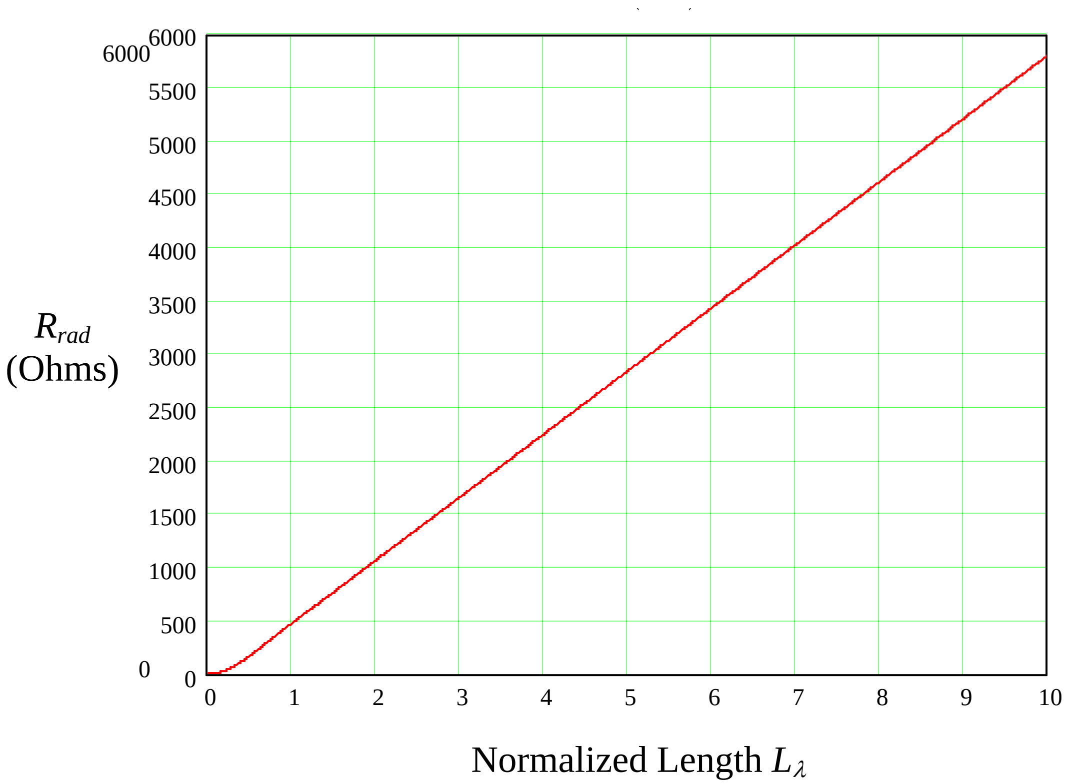

Figure 2a - Radiation resistance for a uniform current dipole over the range $L_\lambda = 0$ to 10.

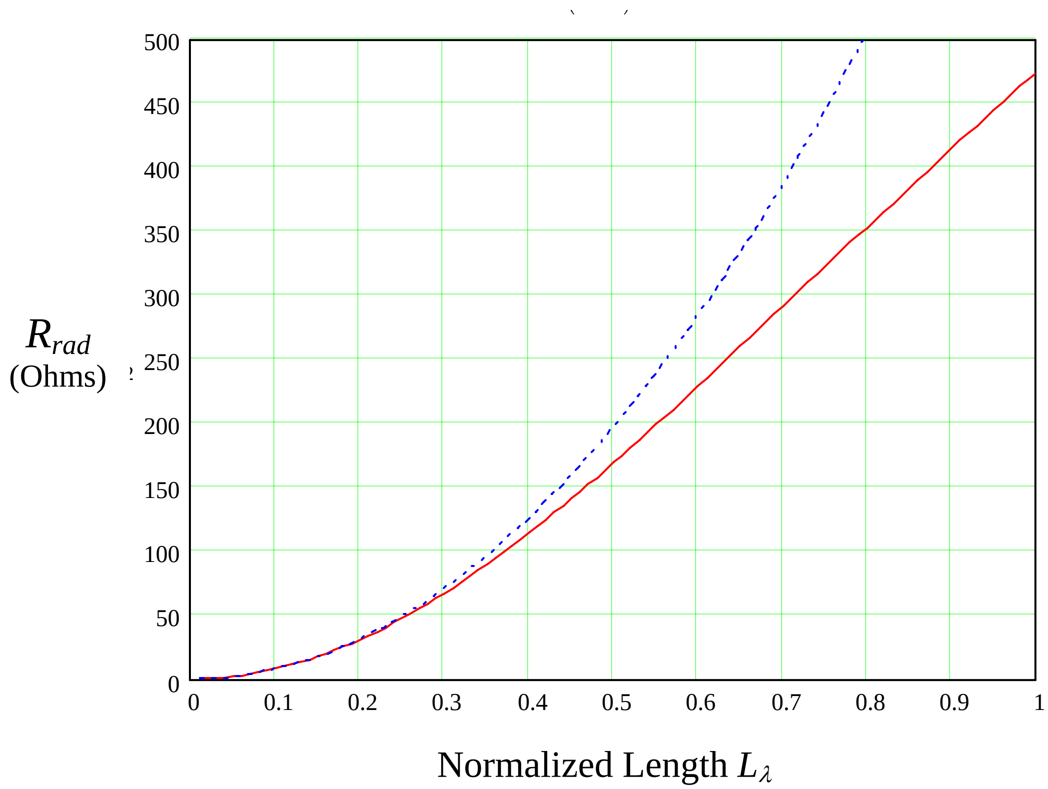

Figure 2b – Radiation resistance for a uniform current dipole (solid curve) over the range $L_\lambda = 0$ to 1. For comparison, the formula $R = 80\pi^2 L_\lambda^2$ is also plotted (dashed line).

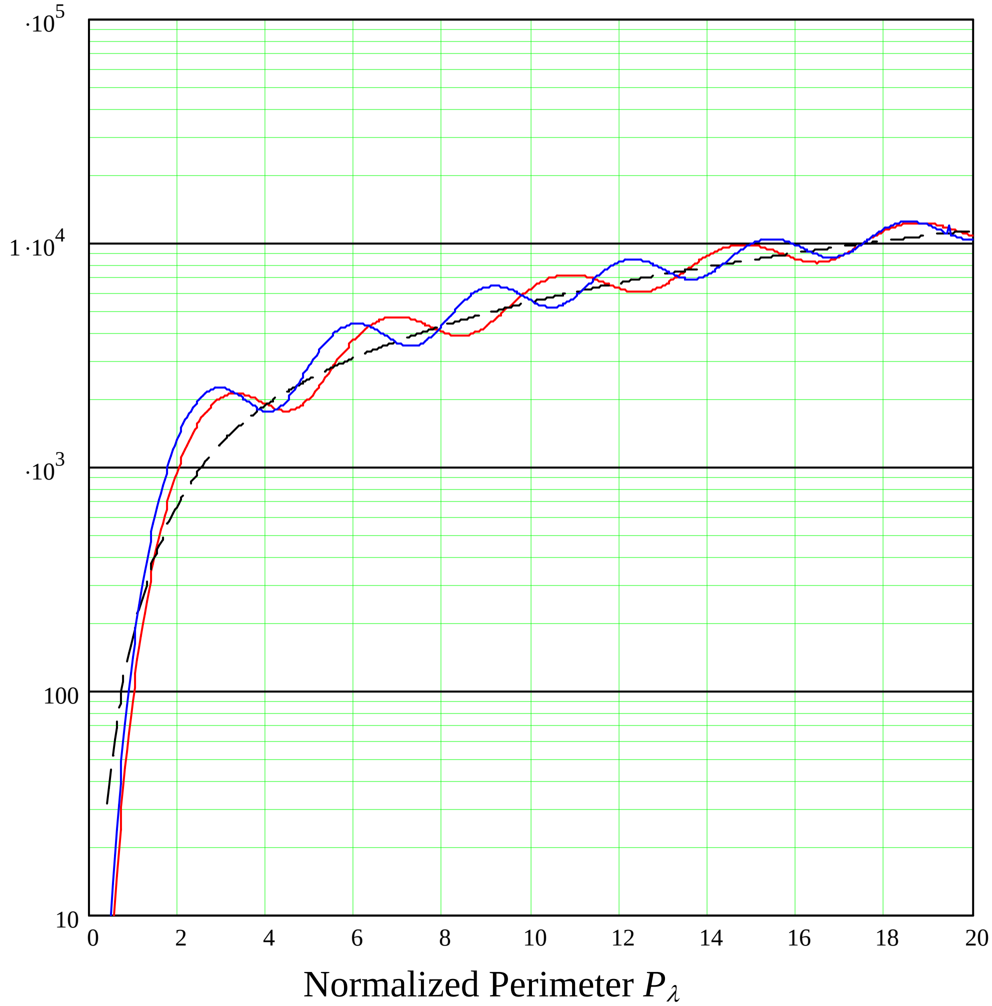

Figure 3 – Calculated radiation resistance of a uniform current square loop (red) compared with the resistance of a uniform current circular loop (blue) by Foster [1]. The dashed curve (black) is four times the self-resistance of a side, $4R_{11}$.

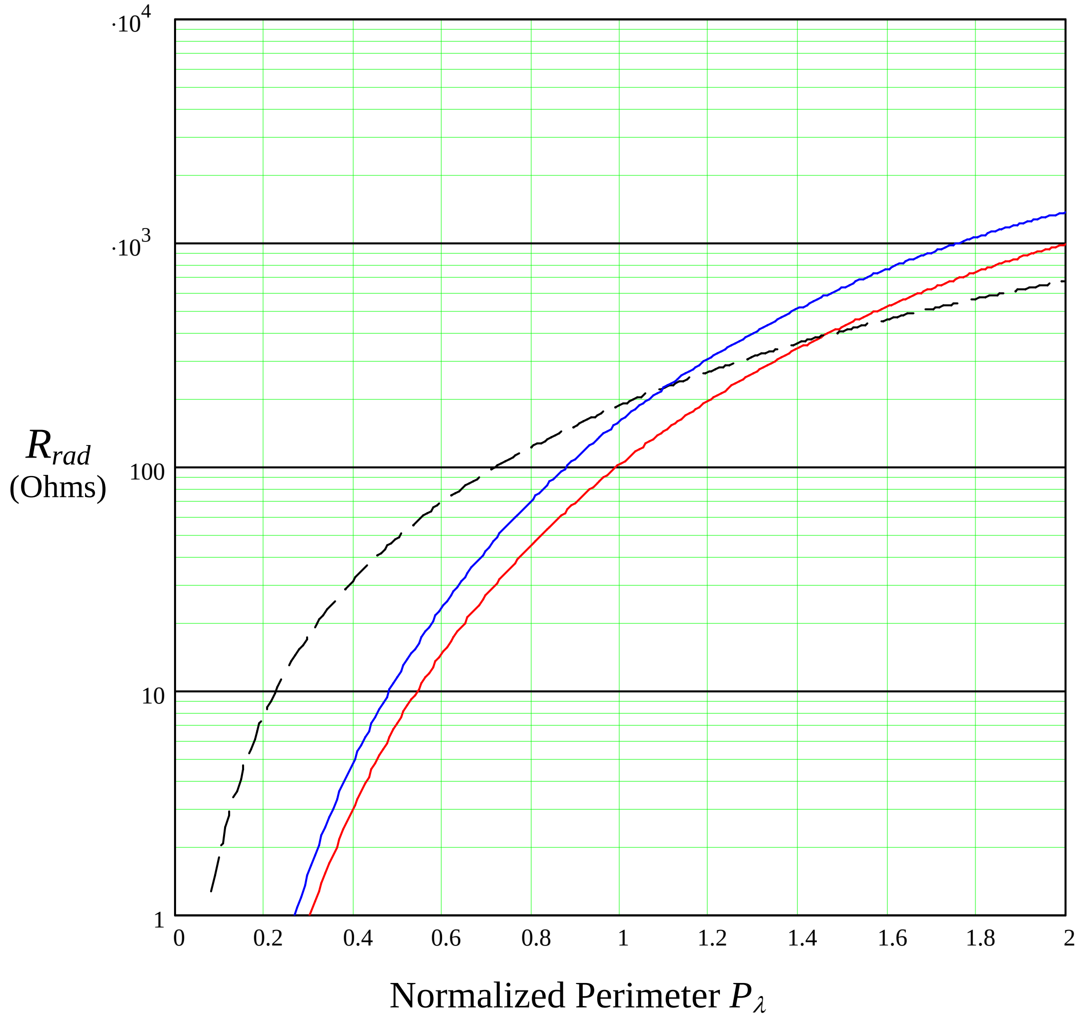

Figure 4 – Radiation resistance for uniform current square loop (red) and circular loop (blue) in the small loop regime. As expected, the curves are in a constant ratio of 1.621. The dashed curve (black) is four times the self-resistance of a side, $4R_{11}$.

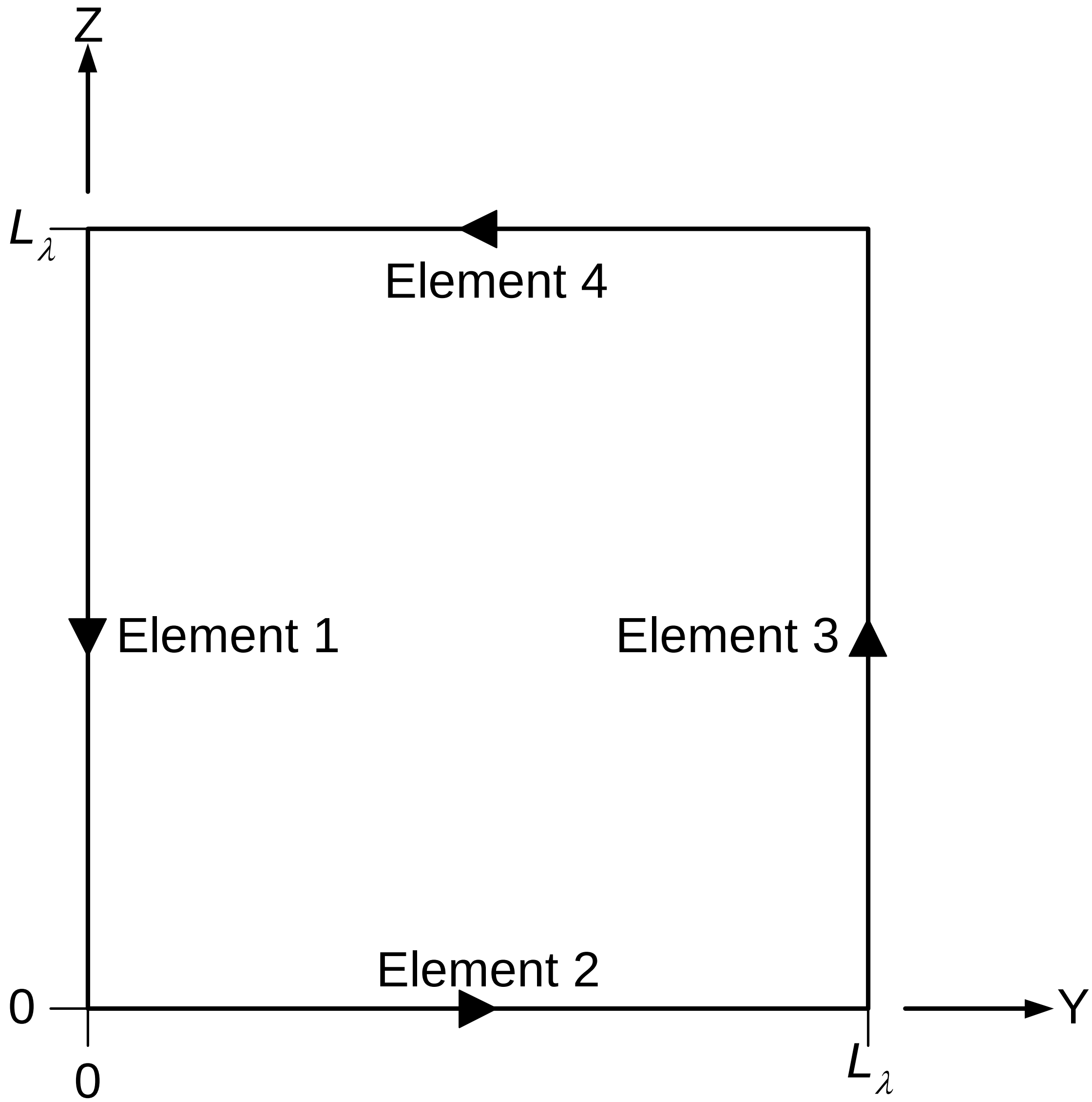
Z
L_λ
Element 4
Element 1
Element 3
Element 2
0
Y
0
L_λ

Figure A1

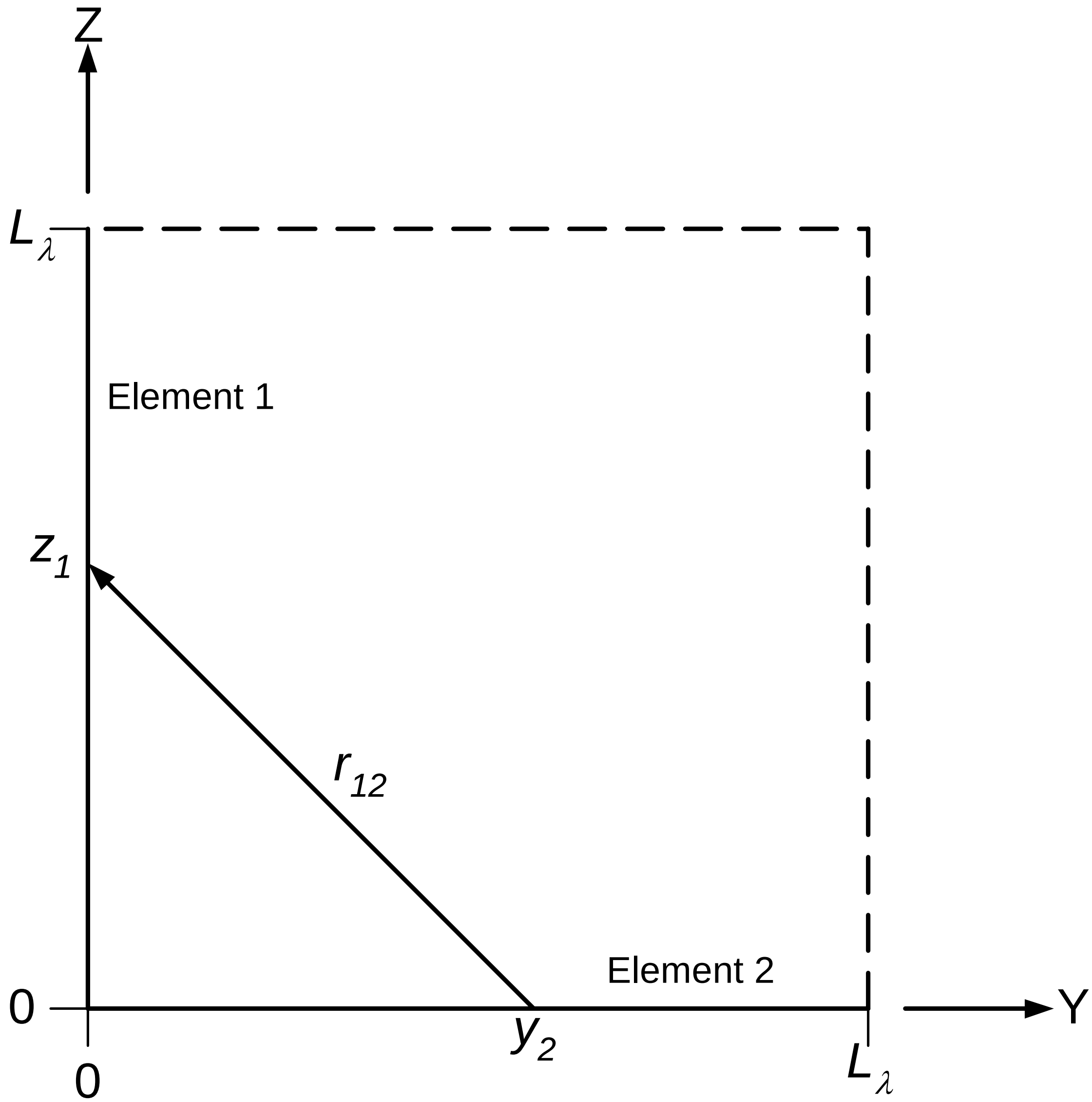

Figure A2

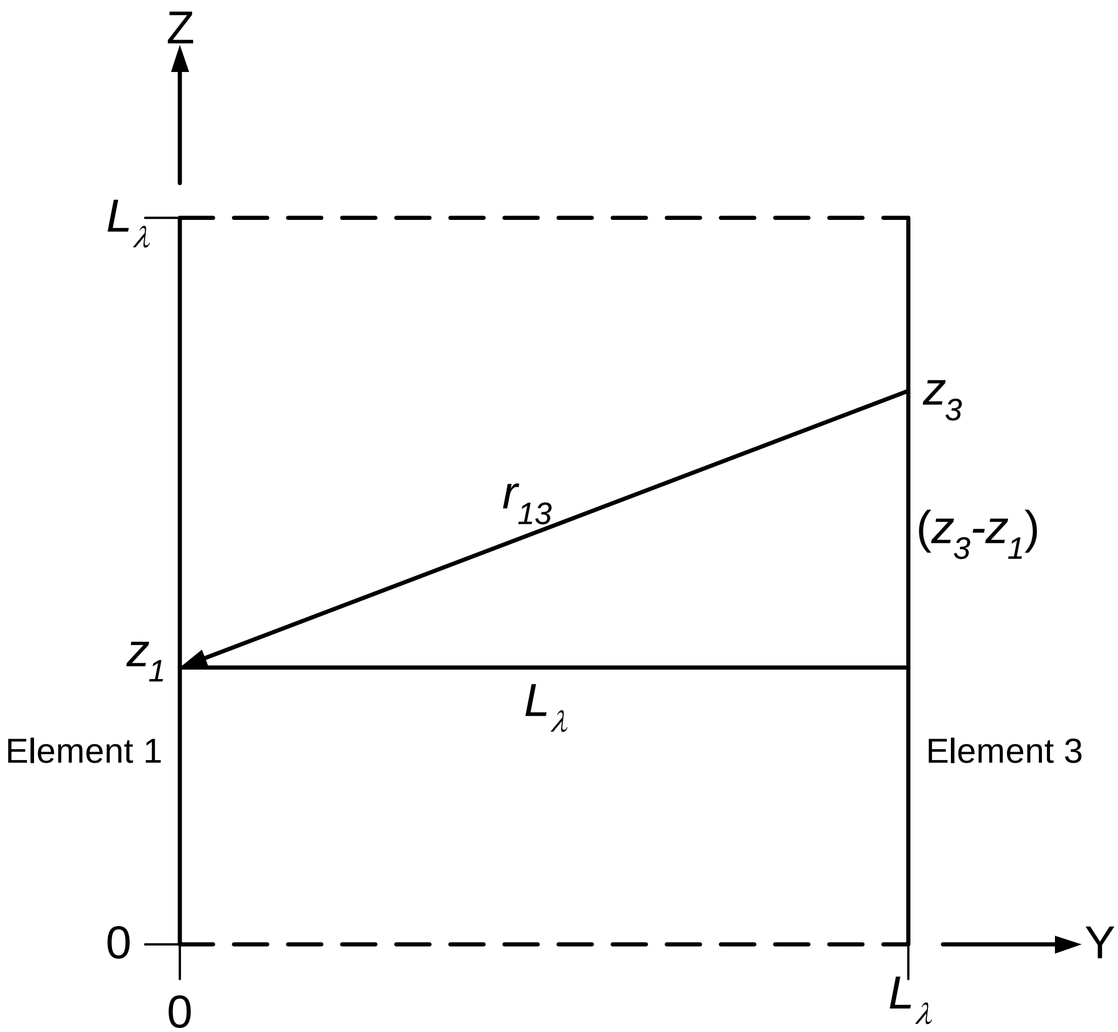

Figure A3

BRIEF BIOGRAPHY OF THE AUTHOR

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